Cascades in helical turbulence

P. D. Ditlevsen[1] and P. Giuliani[1,2]
[1]The Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, DK-2100 Copenhagen O, Denmark
[2]Dipartimento di Fisica and Istituto Nazionale di Fisica della Materia, Università della Calabria, 87036 Rende (CS), Italy
(Received 8 October 1999; published 26 February 2001)

We suggest the existence of a characteristic inner scale ξ for helicity dissipation in a regime of hydrodynamic fully developed turbulence and estimate it on dimensional grounds. This scale is always larger than the Kolmogorov scale η and their ratio η/ξ vanishes in the high Reynolds number limit, so the flow will always be helicity free in the small scales. These ideas are illustrated in a shell model of turbulence.

DOI: 10.1103/PhysRevE.63.036304 PACS number(s): 47.27.Jv

Few exact results regarding fully developed turbulence have been derived as yet. The most celebrated being Kolmogorov's four-fifths law [1]. The four-fifths law is based on the fact that energy, which is an inviscid invariant of the flow, is transferred through the inertial range from the integral scale to the dissipation scale. The four-fifths law, $\langle \delta\mu_{\parallel}(l)^3\rangle = -(4/5)\overline{\varepsilon} l$, states that the third order correlator associated with energy flux equals the mean energy dissipation. As noted recently [2,3] in the case of helical flow a similar relation exists for the transfer of helicity leading to another scaling relation for a third order correlator associated with the flux of helicity, $\langle \delta\mathbf{u}_{\parallel}(l)\cdot[\mathbf{u}_{\perp}(r)\times\mathbf{u}_{\perp}(r+l)]\rangle = (2/15)\overline{\delta} l^2$, where $\overline{\delta}$ is the mean dissipation of helicity. This relation is called the ''two-fifteenths law'' due to the numerical prefactor. This establishes another nontrivial scaling relation for velocity differences in a turbulent helical flow.

The question of cascade of helicity was first discussed by Brissaud *et al.* [4]. Here two possibilities were considered, either there will be coexisting (forward) cascades of energy and helicity or there will be forward cascade of helicity accompanied by an inverse cascade of energy. The latter possibility seems to be ruled out by numerical calculations [5,6]. For the first possibility it was argued on phenomenological grounds that the helicity cascade ''linearly'' follows the energy cascade such that the spectra has the same scaling behavior, $E(k)\sim\overline{\varepsilon}^{2/3}k^{-5/3}$ and $H(k)\sim\overline{\delta\varepsilon^{-1/3}}k^{-5/3}$. This result was supported in a closure calculation (EDQNM) [5] and by direct numerical simulation [7]. The ''linear'' helicity cascade fulfills the (exact) constraint $|H(k)|\leq kE(k)$ but is in conflict with the two-fifteenths law, from which simple dimensional counting would give $H(k)\sim\overline{\delta}^{2/3}k^{-4/3}$ as should be expected in a situation of a ''pure'' helicity cascade. This latter result was obtained by Moiseev and Chkhetiani [8] for turbulence in a stratified medium as the scaling solution in a Hopf-like equation for a characteristic functional. This apparent conflict could be related to the fact that helicity is not a positive quantity that can lead to very different scaling behaviors for even and odd powers of the velocity field [9,10]. We therefore have to be very careful when applying dimensional arguments with respect to the scaling of the helicity spectrum. This is most strongly manifested in the fact that the four-fifths law and the two-fifteenths law have completely different scaling for two correlators that have the same dimensionality but different tensorial structure.

The coexistence of cascades of energy and enstrophy is prohibited for high Reynolds number flow in two-dimensional (2D) turbulence. The reason for this is that the enstrophy dominates at small scales such that the ratio of energy to enstrophy dissipation vanishes for high Reynolds number flow. The inner scale k_Z^{-1} for enstrophy dissipation is determined from the energy spectrum $E(k)\sim k^{-3}$ and the kinematic viscosity ν by $\overline{\zeta}=\nu\int^{k_Z}dk k^4E(k)\sim\nu k_Z^2\Rightarrow k_Z\sim\nu^{-1/2}$, where $\overline{\zeta}$ is the mean dissipation of enstrophy. The energy dissipation is $\overline{\varepsilon}=\nu\int^{k_Z}dk k^2E(k)\sim\nu\ln k_Z\sim -(1/2)\nu\ln\nu\rightarrow 0$ for $\nu\rightarrow 0$. Consequently, energy is cascaded upscale in 2D turbulence.

The existence of simultaneous cascades of energy and helicity is a little surprising because the same type of dimensional argument as for the cascades of energy and enstrophy in 2D turbulence applies. The helicity density is $h=u_i\omega_i$, where $\omega_i=\epsilon_{ijk}\partial_j u_k$ is the vorticity. The mean dissipation of helicity is $D_H=\nu\langle\partial_j u_i\partial_j\omega_i\rangle$. An instructive way of representing this spectrally is to expand the velocity vector $u_i(\mathbf{k})$ in a basis of ''helical modes'' [11]. The helical modes $\mathbf{h}_{\pm}$ are simply the (complex) eigenvectors of the curl operator, $i\mathbf{k}\times\mathbf{h}_{\pm}=\pm k\mathbf{h}_{\pm}$.

Using incompressibility, $\mathbf{k}\cdot\mathbf{u}(\mathbf{k})=0$, we have $\mathbf{u}(\mathbf{k})=u_+(\mathbf{k})\mathbf{h}_+ + u_-(\mathbf{k})\mathbf{h}_-$ and the energy and helicity in the mode $\mathbf{u}(\mathbf{k})$ are $E(\mathbf{k})=\mathbf{u}(\mathbf{k})\cdot\mathbf{u}(\mathbf{k})^*/2=(|u_+(\mathbf{k})|^2+|u_-(\mathbf{k})|^2)/2$ and $H(\mathbf{k})=\mathbf{u}(\mathbf{k})\cdot\boldsymbol{\omega}(\mathbf{k})^*=k(|u_+(\mathbf{k})|^2-|u_-(\mathbf{k})|^2)$.

The spectral energy density can then be separated into the densities of modes of positive and negative helicities $E(k)=E_+(k)+E_-(k)$, and we finally arrive at the expression for the spectral helicity dissipation,

$$D_H=2\nu\int^{k_E}dk k^3[E_+(k)-E_-(k)]\sim\nu k_E^{7/3}\sim\nu^{-3/4}, \quad (1)$$

where $k_E^{-1}=\eta$ is the Kolmogorov scale and we have used $E(k)\sim k^{-5/3}$ and $k_E\sim\nu^{-3/4}$. The scaling derived holds for the helical modes of either sign separately and would hold in a helical fluid for the total helicity unless there is a detailed balance between the helicities of opposite signs. This means that for high Reynolds number flow, the dissipation of helicity will grow as $Re^{3/4}$. Since the mean dissipations of energy $\overline{\varepsilon}$ and helicity $\overline{\delta}$ are determined by the integral scale forcing, the growth of helicity dissipation with Reynolds number is

 ©2001 The American Physical Society

apparently in conflict with the assumption of a constant energy and helicity dissipation in the limit of vanishing viscosity. This is not a true problem because helicity is nonpositive and the integrand in Eq. (1) can have either sign. So in the high Reynolds number limit, there must either be a detailed balance between dissipation of positive and negative helicity or the energy cascade is blocked [12]. In the rather artificial case of a shell model where only one sign of helicity is dissipated by hyperviscosity, the energy cascade is indeed prevented altogether similar to the case of forward energy cascade of energy in 2D turbulence [13].

We suggest that in a helical flow ($\overline{\delta}\neq 0$) the dissipation of helicity defines a scale ξ different from the Kolmogorov scale η.

Following K41, the Kolmogorov scale η is obtained from $\overline{\varepsilon}\sim \delta u_\eta^3/\eta \sim \nu \delta u_\eta^2/\eta^2 \Rightarrow \eta \sim (\nu^3/\overline{\varepsilon})^{1/4}$, where δu_l is a typical variation of the velocity over a scale l. The inner scale ξ for dissipation of helicity is defined as the scale where the helicity dissipation is of same order as the spectral helicity flux. With dimensional counting we have $\overline{\delta}\sim \nu \delta u_\xi^2/\xi^3$ and using K41, $\delta u_l \sim (l\overline{\varepsilon})^{1/3}$, we obtain

$$\xi \sim (\nu^3 \overline{\varepsilon}^2/\overline{\delta}^3)^{1/7}. \tag{2}$$

Now it is clear why Eq. (1) leads to a wrong conclusion for the mean dissipation of the helicity $\overline{\delta}$. The integral will not be dominated by contributions from k_E but contributions from $k_H = \xi^{-1}$

$$D_H = \overline{\delta} \sim \nu k_H^{7/3} \Rightarrow k_H \sim \nu^{-3/7}. \tag{3}$$

The ratio of the two inner scales is then $(\eta/\xi) = (k_H/k_E) \sim \nu^{-3/7+3/4} = \nu^{9/28} \to 0$ for $\nu \to 0$. Thus for high Reynolds number helical flow the small scales will always be nonhelical.

The reason for the flow to be nonhelical on small scales is different from the reason why the flow tends to be isotropic on small scales even though the integral scale is nonisotropic. The reason for the small scales to be isotropic is that the structure functions associated with the nonisotropic sectors scale with scaling exponents that are larger than those of the isotropic sector and thus becomes subleading for the flow at small scales independent of the dissipation [14].

The physical picture for fully developed helical turbulence is that $\overline{\delta}$ and $\overline{\varepsilon}$ are solely determined by the forcing in the integral scale. There will then be an inertial range with coexisting cascades of energy and helicity with third order structure functions determined by the four-fifths and the two-fifteenths laws. This is followed by an inertial range between ξ and η corresponding to nonhelical turbulence, where the dissipation of positive and negative helicity vortices balance and the two-fifteenths law is not applicable.

The inner scale for helicity has not been identified in numerical simulations where hyperviscosity is used to extend the inertial range [7]. For this to be done, one would need to apply normal viscosity in a simulation. The scale could, in principle, be determined in experiments by measurements of the third order correlator $\langle \delta \mathbf{u}_\parallel(l)\cdot[\mathbf{u}_\perp(r)\times \mathbf{u}_\perp(r+l)]\rangle$ to determine the scaling range in accordance with the two-fifteenth law. A good candidate for measurements would be the atmospheric boundary layer, where helicity is pumped into the system by the Earth's rotation.

In order to test these ideas in a model system we investigate the role of helicity and the structure of the helicity transfer in a shell model.

Shell models are toy models of turbulence, which by construction have second order inviscid invariants similar to energy and helicity in 3D turbulence. Shell models can be investigated numerically for high Reynolds numbers, in contrast to the Navier-Stokes equation, and high order statistics are easily accessible. Shell models lack any spatial structures, so we stress that only certain aspects of turbulent cascades have meaningful analogies in shell models. This should especially be kept in mind when studying helicity which is intimately linked to spatial structures, and the dissipation of helicity to reconnection of vortex tubes [12]. So the following only concerns the spectral aspects of the helicity and energy cascades.

The most well-studied shell model, the GOY model [15], is defined from the governing equation,

$$\dot{u}_n = ik_n\left(u_{n+2}u_{n+1} - \frac{\epsilon}{\lambda}u_{n+1}u_{n-1} + \frac{\epsilon-1}{\lambda^2}u_{n-1}u_{n-2}\right)^* - \nu k_n^2 u_n + f_n \tag{4}$$

with $n=1,2,\ldots$ where the u_n's are the complex shell velocities. The wave numbers are defined as $k_n=\lambda^n$, where λ is the shell spacing. The second and third terms are dissipation and forcing. The model has two inviscid invariants; energy $E=\sum_n E_n = \sum_n |u_n|^2/2$ and "helicity" $H=\sum_n H_n = \sum_n (\epsilon-1)^{-n}|u_n|^2$. The model has two free parameters, λ and ϵ. The "helicity" only has the correct dimension of helicity if $|\epsilon-1|^{-n}=k_n \Rightarrow 1/(1-\epsilon)=\lambda$. In this work we use the standard parameters $(\epsilon,\lambda)=(1/2,2)$ for the GOY model.

The energy flux is defined in the usual way as $\Pi_n^E = -d/dt|_{nl}(\sum_{m=1}^n E_m)$ where $d/dt|_{nl}$ is the time rate of change due to the nonlinear term in Eq. (4). The helicity flux Π_n^H is defined similarly. By a simple algebra we have the following expression for the fluxes:

$$\langle \Pi_n^E \rangle = (\epsilon-1)\Delta_n - \Delta_{n+1} = \overline{\varepsilon}, \tag{5}$$

$$\langle \Pi_n^H \rangle = 2(-1)^{n+1}k_n(\Delta_{n+1}-\Delta_n) = \overline{\delta}, \tag{6}$$

where $\Delta_n = k_{n-1}\mathrm{Im}\langle u_{n-1}u_n u_{n+1}\rangle$, $\overline{\varepsilon}$ and $\overline{\delta}$ are the mean dissipations of energy and helicity, respectively. The first equalities hold without averaging as well. These equations are the shell model equivalents of the four-fifths and the two-fifteenths law.

In the shell model we have $\overline{\varepsilon} = \sum_n \mathrm{Re}\langle f_n u_n^* \rangle$ and $\overline{\delta} = 2\sum_n (-1)^n k_n \mathrm{Re}\langle f_n u_n^*\rangle$. So in complete analogy with Navier-Stokes (N-S) turbulence we have $|\overline{\delta}| \leq 2k_f \overline{\varepsilon}$, where k_f is a wave number such that $f_k=0$ for $k>k_f$ provided there is a non-negative mean energy injection for all wave numbers $k\leq k_f$. The forcing can be chosen in many ways. A

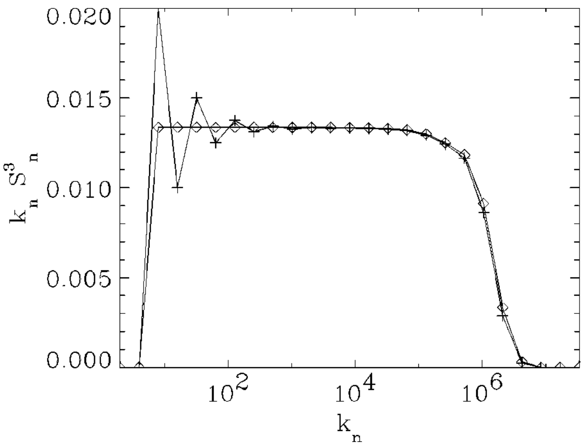

FIG. 1. The third order structure function S_n^3 in the cases $\overline{\delta}>0$ (crosses) and $\overline{\delta}=0$ (diamonds). In the case of helicity free forcing the modulus-2 oscillation disappears. In the two runs we have 25 shells, $\nu=10^{-9}$, $f_n=0.01(1+i)(\delta_{n,2}/u_2^*-A\delta_{n,3}/2u_3^*)$ with $A=0,1$ respectively.

natural choice is $f_n=f_n^0/u_n^*$, where f_n^0 is independent on the shell velocities. Then we have, $\overline{\varepsilon}=\Sigma_{n<n_f}f_n^0$ and $\overline{\delta}=\Sigma_{n<n_f}(-1)^n k_n f_n^0$, where n_f indicates the end of the integral scale. By choosing the coefficients, stochastic or deterministic functions of time, this last sum can vanish identically, which is referred to as helicity free forcing. As traditional for shell models the third order correlation function, $S_n^3 \equiv -\mathrm{Im}\langle u_{n-1}u_n u_{n+1}\rangle = -\Delta_n/k_{n-1}$ is obtained from Eqs. (5) and (6),

$$k_n S_n^3 = \frac{2}{(2-\epsilon)}[\overline{\varepsilon}-(-1)^n\overline{\delta}/k_n]. \qquad (7)$$

The last term in the parenthesis is subleading with period-2 oscillations. When $\overline{\delta}=0$, the subleading term disappears and the scaling from the equivalent of the four-fifths law (5) is obtained, Fig. 1. The simulations are performed with the

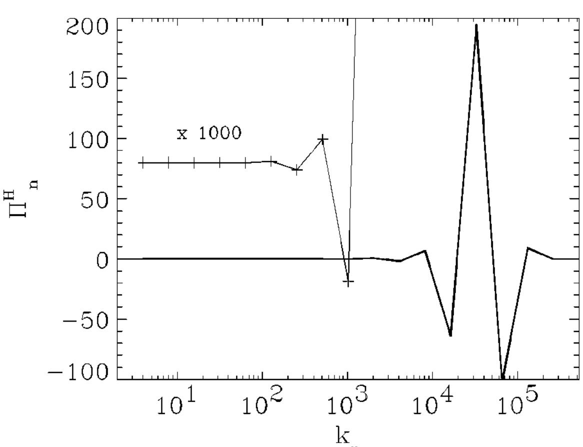

FIG. 2. The helicity flux $\langle \Pi_n^H\rangle$ in the case $\overline{\delta}>0$. The same curve is multiplied by 1000 and overplotted in order to see the inertial range. The period-2 oscillation in the helicity transfer comes from the helicity dissipation.

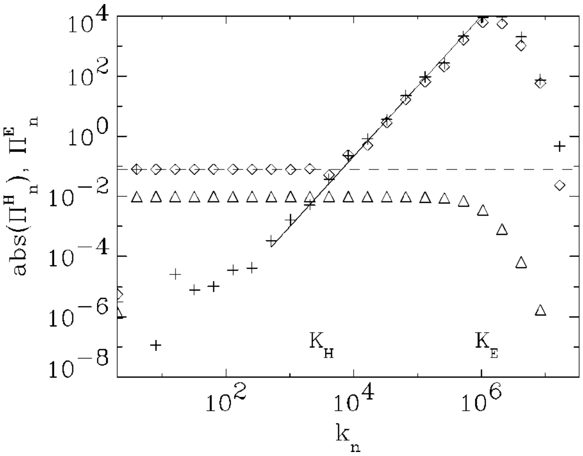

FIG. 3. The absolute values of the helicity flux $|\langle \Pi_n^H\rangle|$ (diamonds) show a crossover from the inertial range for helicity to the range where the helicity is dissipated. The line has a slope of 7/3 indicating the helicity dissipation. The dashed lines indicate the helicity input $\overline{\delta}$. The crosses are the helicity flux in the case $\overline{\delta}=0$ where there is no inertial range and K_H coincides with the integral scale. The triangles are the energy flux $\langle \Pi_n^E\rangle$.

forcing $f_2^0=10^{-2}(1+i)$ and $f_3^0=-Af_2^0/\lambda$ with $A=1$ and $A=0$, corresponding to $(\overline{\varepsilon},\overline{\delta})=(0.01,0)$ (diamonds) and $(\overline{\varepsilon},\overline{\delta})=(0.01,0.08)$ (crosses).

Helicity is not positive and is dissipated with opposite signs for odd and even shells. If we consider the third order structure function associated with the helicity transfer as defined by Eq. (6) we see (Fig. 2) period-2 oscillations growing with n. This period-2 oscillation is due to the dissipation. The helicity flux is

$$\langle \Pi_n^H\rangle = \overline{\delta} - \langle D_n^H\rangle, \qquad (8)$$

where D_n^H is the helicity dissipation at shells $m \leq n$,

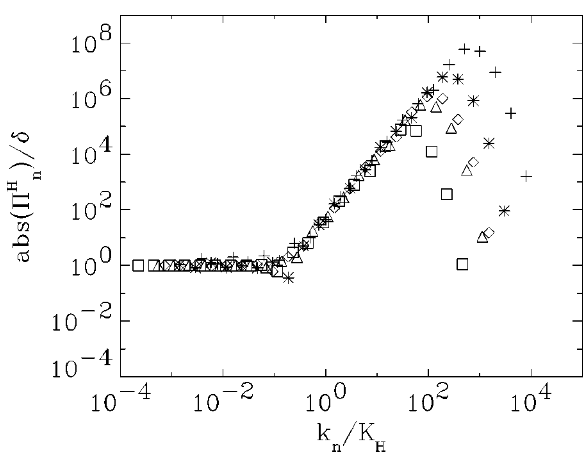

FIG. 4. Five simulations with constant viscosity $\nu=10^{-9}$, constant energy input $\overline{\varepsilon}=0.01$, and varying helicity input $\overline{\delta}=(0.0001,0.001,0.005,0.01,0.08)$ are shown. The absolute values of the helicity flux $|\langle \Pi_n^H\rangle|$ divided by $\overline{\delta}$ is plotted against the wave number divided by $K_H=(\nu^3\overline{\varepsilon}^2/\overline{\delta}^3)^{-1/7}$, which is obtained from Eq. (2) neglecting $O(1)$ constants. A clear data collapse is seen.

$$D_n^H = 2\nu \sum_{m=1}^{n} (-1)^m k_m^3 |u_m|^2. \tag{9}$$

In the inertial range for energy transfer we have the Kolmogorov scaling $u_n \sim k_n^{-1/3}$, so the helicity dissipation can be estimated,

$$D_n^H \sim 2\nu \sum_{m=1}^{n} (-1)^m k_m^{7/3} \sim \lambda^{7/3} \frac{(-1)^n \lambda^{7n/3} - 1}{\lambda^{7/3}+1} \sim (-1)^n k_n^{7/3}. \tag{10}$$

This is the shell model equivalent of Eq. (1) if n is at the Kolmogorov scale. Figure 3 shows $|\langle \Pi_n^H \rangle|$ and $\langle \Pi_n^E \rangle$ as functions of wave number. The scaling (10) of the helicity dissipation is the straight line, the horizontal dashed line is $\overline{\delta}$. The inertial range for helicity transfer is to the left of the crossing of the two lines. The crossing is the inner scale for helicity transfer K_H, which does not coincide with the Kolmogorov scale K_E. The ''pile-up'' for k larger than K_H was earlier interpreted as a bottleneck effect [16]. It is a balance between positive and negative helicity dissipation and does not contribute to the dissipation of the injected helicity.

In order to verify the scaling relation between K_H and $\overline{\delta}$, we performed a set of simulations with constant energy input $\overline{\varepsilon} = 0.01$ and varying helicity input $\overline{\delta} = (0.0001, 0.001, 0.005, 0.01, 0.08)$. In Fig. 4 the spectra of the absolute value of the helicity transfer normalized with $\overline{\delta}$ are plotted versus wave number normalized with K_H. K_H is in each case calculated from Eq. (2), and a clear data collapse is seen.

In summary, an inner scale for helicity dissipation has been identified. This scale is always larger than the Kolmogorov scale. Thus there exist two inertial ranges in helical turbulence—a range smaller than ξ with coexisting cascades of energy and helicity where both the four-fifths and the two-fifteenths law applies, and a range between ξ and η, where the flow is nonhelical and only the four-fifths law applies. These findings could potentially be verified in observations or in direct numerical simulations of helical turbulence.

We would like to thank Uriel Frisch, Marcel Lesieur, Nicolaj Antonov, and Otto Chkhetiani for useful comments.

[1] U. Frisch, *Turbulence, The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).
[2] V.S. L'vov *et al.*, chao-dyn/9705016 (unpublished).
[3] O. Chkhetiani, Pisma Zh. Éksp. Teor. Fiz. **63**, 768 (1996) [JETP Lett. **63**, 808 (1996)].
[4] A. Brissaud *et al.*, Phys. Fluids **16**, 1366 (1973).
[5] J.C. André and M. Lesieur, J. Fluid Mech. **81**, 187 (1977).
[6] M. Lesieur, *Turbulence in Fluids*, 3rd ed. (Kluwer Academic, Dordrecht, 1997).
[7] V. Borue and S.A. Orszag, Phys. Rev. E **55**, 7005 (1997).
[8] S.S. Moiseev and O.G. Chkhetiani, Zh. Éksp. Teor. Fiz. **110**, 357 (1996) [JETP **83**, 192 (1996)].
[9] L. Biferale, D. Pierotti, and F. Toschi, Phys. Rev. E **57**, R2515 (1998).
[10] P.D. Ditlevsen and P. Giuliani, Physica A **280**, 69 (2000).
[11] F. Waleffe, Phys. Fluids A **4**, 350 (1992).
[12] E. Levich, L. Shtilman, and A.V. Tur, Physica A **176**, 241 (1991).
[13] P.D. Ditlevsen, Phys. Fluids **9**, 1482 (1997).
[14] I. Arad *et al.*, Phys. Rev. Lett. **82**, 5040 (1999).
[15] E.B. Gledzer, Dokl. Akad. Nauk **209**, 1046 (1973) [Sov. Phys. Dokl. **18**, 216 (1973)].
[16] L. Biferale, D. Pierotti, and F. Toschi, J. Phys. IV **8**, Pr6 (1998).